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# An instability of internal gravity waves

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When water is slightly stratified, internal gravity waves are considerably shorter than surface waves of comparable frequency. Here, this fact is exploited in demonstrating that an internal wave is unstable when it forms part of a resonant triad with a surface wave and another internal wave whose wavenumber is approximately equal to that of the original internal wave. It is suggested that in a system where there are two classes of waves of comparable frequencies but greatly differing wavelengths the short waves may be expected to generate long waves by this mechanism.

## 1. Introduction

In the last decade a large body of knowledge has been gathered concerning weakly nonlinear interactions between trains of gravity waves. These interactions are of particular importance when the frequencies $\omega_j$ and wavenumbers $\mathbf{k}_j$ satisfy, or nearly satisfy, the resonance conditions

$$\sum_j \pm (\omega_j, \mathbf{k}_j) = 0, \qquad (1)$$

where $\omega_j$ and $\mathbf{k}_j$ are connected by a dispersion relation. Despite the invention of ingeneous geometrical constructions, it is often difficult to determine whether such sets of waves exist and more difficult to determine the members of such sets (Ball 1964; Thorpe 1966; Simmons 1969). However, there are a few special circumstances in which the resonance conditions present no difficulty. For example, four waves of the same class with almost equal wavenumbers can be combined in such a way that (1) is almost satisfied (Benjamin & Feir 1967). Again, in a system where there are two classes of waves of comparable wavenumbers but vastly different frequencies, two high frequency waves of almost identical frequency but distinct wavenumbers $\mathbf{k}_1$ and $\mathbf{k}_2$ can form a resonant triad with a low frequency wave of wavenumber $\mathbf{k}_1 - \mathbf{k}_2$ (Phillips 1966, p. 172).

For progressive waves there is considerable duality between the spatial and temporal variations (i.e. for most purposes $x$ and $t$ are interchangeable), and in this paper we shall study a class of wave interactions which can be considered as being the dual of the final example described above. Thus we are concerned with a system where there are two classes of waves of comparable frequencies but vastly different wavenumbers and hence two short waves of almost identical wavenumbers but distinct frequencies $\omega_1$ and $\omega_2$ can form a resonant triad with a long wave of frequency $\omega_1 - \omega_2$.

The system which will be examined in detail is slightly stratified water where the short and long waves are internal and surface waves respectively. We shall be particularly concerned with the role of the resonant triad as an instability mechanism for internal waves. It is clear that the minimum amplitude for which there is an instability will depend strongly upon the attenuation rates of the three interacting waves. Here attenuation will be allowed for solely through the use of empirical factors, which are envisaged as giving the correct decay rates when there are no interactions. For waves of a fixed wavenumber it is known that dissipation produces frequency shifts of the same small order as the attenuation rates (Le Blond 1966). However, it can readily be shown that all the results in this paper are extremely insensitive to the frequency shifts but not to the decay rates.

## 2. Interaction equations

The equations of motion and boundary conditions for small amplitude motions of an inviscid, slightly stratified, incompressible fluid can be reduced to the form (Phillips 1966, p. 162)

$$\left.\begin{aligned} \frac{\partial^2}{\partial t^2}\left(\nabla_h^2+\frac{\partial^2}{\partial z^2}\right)w+N^2\nabla_h^2 w &= Q, \\ \frac{\partial^2}{\partial t^2}\left(\frac{\partial w}{\partial z}\right)-g\nabla_h^2 w = R \quad &\text{at} \quad z=0, \\ w=0 \quad \text{at} \quad z&=-D. \end{aligned}\right\} \tag{2}$$

Here $\nabla_n$ denotes the horizontal gradient operator $(\partial/\partial x, \partial/\partial y, 0)$, $z$ is the vertical co-ordinate, $N$ the Brunt–Väisälä frequency, $D$ the depth of the water and $Q, R$ are nonlinear terms of considerable complexity (see the appendix).

Let $W(z;\mathbf{m},n)$ be a solution to the eigenvalue problem

$$\frac{d^2W}{dz^2}+\left(\frac{N^2m^2}{n^2}-m^2\right)W=0,$$

$$\frac{dW}{dz}-\frac{gm^2}{n^2}W=0 \quad \text{at} \quad z=0,$$

$$W=0 \quad \text{at} \quad z=-D,$$

where, without loss of generality, we shall assume that $W$ is normalized by the condition

$$\frac{1}{D}\int_{-D}^{0} W^2 dz = 1.$$

For each eigenvalue pair $\mathbf{m}, n$ it is straightforward to derive a single equation from the system (2): i.e.

$$\int_{-D}^{0} W\left\{\left(\frac{\partial^2}{\partial t^2}\nabla_h^2-n^2m^2\right)w+N^2(\nabla_h^2+m^2)w-Q\right\}dz + [W\{g(\nabla_h^2+m^2)w+R\}]_{z=0}=0. \tag{3}$$

For the instability mechanism under consideration it suffices to represent the vertical component of velocity by

$$\begin{aligned} w = {} & W(z;\mathbf{k},\omega_1)\{\Phi_1 \exp[i\omega_1 t + i\mathbf{k}.\mathbf{x}] + *\} \\ & + W(z;\mathbf{k},\omega_2)\{\Phi_2 \exp[i\omega_2 t + i(\mathbf{k}-(1+\delta)\boldsymbol{\kappa}).\mathbf{x}] + *\} \\ & + W(z;\boldsymbol{\kappa},\omega_1-\omega_2)\{\Phi_3 \exp[i(\omega_1-\omega_2)t + i(1+\delta)\boldsymbol{\kappa}.\mathbf{x}] + *\} + w', \end{aligned} \tag{4a}$$

where $\Phi_1$, $\Phi_2$, $\Phi_3$ are the slowly varying amplitudes of the three interacting waves, $\mathbf{k}$ is the wavenumber of the original internal wave, $\boldsymbol{\kappa}$ is the wavenumber of an unforced surface wave of frequency $\omega_1-\omega_2$, $\delta$ is a small number which allows for the fact that the surface wave is a forced wave, $*$ denotes the complex conjugate of explicitly written terms and $w'$ indicates other contributions to $w$ which play no role in the interactions. In a like manner we represent $Q$ and $R$ by the expressions

$$\begin{aligned} \{Q_1 \exp[i\omega_1 t + i\mathbf{k}.\mathbf{x}] + Q_2 \exp[i\omega_2 t + i(\mathbf{k}-(1+\delta)\boldsymbol{\kappa}).\mathbf{x}] \\ + Q_3 \exp[i(\omega_1-\omega_2)t + i(1+\delta)\boldsymbol{\kappa}.\mathbf{x}] + *\} + Q' \end{aligned} \tag{4b}$$

and

$$\begin{aligned} \{R_1 \exp[i\omega_1 t + i\mathbf{k}.\mathbf{x}] + R_2 \exp[i\omega_2 t + i(\mathbf{k}-(1+\delta)\boldsymbol{\kappa}).\mathbf{x}] \\ + R_3 \exp[i(\omega_1-\omega_2)t + i(1+\delta)\boldsymbol{\kappa}.\mathbf{x}] + *\} + R'. \end{aligned} \tag{4c}$$

We note that the form of (4*a*) implies that $\omega_2$ and $\boldsymbol{\kappa}$ are determined through the explicit equations

$$\omega_2 = \omega_2(\mathbf{k}) \quad \text{and} \quad \boldsymbol{\kappa} = \boldsymbol{\kappa}(\omega_1-\omega_2)$$

rather than through the more complicated exact implicit equations

$$\omega_2 = \omega_2(\mathbf{k}-\boldsymbol{\kappa}) \quad \text{and} \quad \boldsymbol{\kappa} = \boldsymbol{\kappa}(\omega_1-\omega_2).$$

However, in this paper we shall repeatedly make use of the fact that $\boldsymbol{\kappa}/k$ is very small, which introduces errors of the same order as the errors involved when using explicit expressions for $\omega_2$ and $\boldsymbol{\kappa}$.

By substituting the representation (4) into (3) and then extracting the $\exp[i\omega_1 t + \mathbf{k}.\mathbf{x}]$ coefficient of the $\mathbf{k}, \omega_1$ equation we obtain the relationship

$$\begin{aligned} \frac{\partial \Phi_1}{\partial t} + \omega_1 \left[1 - \frac{1}{D}\int_{-D}^{0} \frac{N^2}{\omega_1^2} W_1^2 dz - \frac{g W_1^2}{D\omega_1^2}\bigg|_{z=0}\right] \frac{\mathbf{k}.\nabla_h \Phi_1}{k^2} + \nu_1 \Phi_1 \\ = \frac{i}{2\omega_1 k^2 D}\left[\int_{-D}^{0} W_1 Q_1 dz - R_1 W_1\bigg|_{z=0}\right], \end{aligned} \tag{5a}$$

where second and higher derivatives of $\Phi_1$ have been neglected on the assumption that the wave amplitude varies slowly and the term $\nu_1 \Phi_1$ is an empirical representation of the effects of dissipation. The corresponding equations for $\Phi_2$ and $\Phi_3$ are

$$\begin{aligned} \frac{\partial \Phi_2}{\partial t} + \omega_2 \left[1 - \frac{1}{D}\int_{-D}^{0} \frac{N^2 W_2^2}{\omega_2^2} dz - \frac{g W_2^2}{D\omega_2^2}\bigg|_{z=0}\right] \left[\frac{\mathbf{k}.\nabla_h \Phi_2}{k^2} - \frac{i\boldsymbol{\kappa}.\mathbf{k}\Phi_2}{k^2}\right] + \nu_2 \Phi_2 \\ = \frac{i}{2\omega_2 k^2 D}\left[\int_{-D}^{0} W_2 Q_2 dz - R_2 W_2\bigg|_{z=0}\right] \end{aligned} \tag{5b}$$

and

$$\frac{\partial \Phi_3}{\partial t}+(\omega_1-\omega_2)\left[1-\frac{1}{D}\int_{-D}^{0}\frac{N^2W_3^2}{(\omega_1-\omega_2)^2}dz-\frac{gW_3^2}{(\omega_1-\omega_2)^2}\bigg|_{z=0}\right]\left[\frac{\kappa.\nabla_h\Phi_3}{\kappa^2}+i\delta\Phi_3\right]+\nu_3\Phi_3$$
$$=\frac{i}{2(\omega_1-\omega_2)\kappa^2D}\left[\int_{-D}^{0}W_3Q_3dz-R_3W_3\bigg|_{z=0}\right], \qquad (5c)$$

where considerable use has been made of the facts that $\delta$ and $\kappa/k$ are very small.

## 3. Instability theory

Equations (5), with the nonlinear coupling terms as given in the appendix, can be written in the form

$$\frac{\partial \Phi_1}{\partial t}+\mathbf{c}_1.\nabla_h\Phi_1+\nu_1\Phi_1 = B+\alpha_1\Phi_2\Phi_3, \qquad (6a)$$

$$\frac{\partial \Phi_2}{\partial t}+\mathbf{c}_2.\nabla_h\Phi_2+\nu_2\Phi_2-i\mathbf{c}_2.\boldsymbol{\kappa}\Phi_2 = \alpha_2\Phi_1\Phi_3^*, \qquad (6b)$$

$$\frac{\partial \Phi_3}{\partial t}+\mathbf{c}_3.\nabla_h\Phi_3+\nu_3\Phi_3+i\delta\mathbf{c}_3.\boldsymbol{\kappa}\Phi_3 = \alpha_3\Phi_1\Phi_2^*, \qquad (6c)$$

where the $\mathbf{c}_j$ are the group velocities of the three waves, the $\alpha_j$ are coupling coefficients and $B$ represents the means whereby the original internal wave is driven (possibly through an instability of a short surface wave). Other examples of the mechanism under consideration will lead to equations of the same form as (6).

In considering the onset of instability it is convenient to assume that the wave amplitude $\Phi_1$ of the original wave is constant. Such a state could be achieved in practice if the forcing term $B$ were constant. In such a situation equations (6$b$) and (6$c$) are coupled linear constant-coefficient equations. Solutions proportional to $\exp[ist+i\mathbf{m}.\mathbf{x}]$ may be sought, and this leads to the characteristic equation

$$(s+\mathbf{c}_2.(\mathbf{m}-\boldsymbol{\kappa})-i\nu_2)(s+\mathbf{c}_3.(\mathbf{m}+\delta\boldsymbol{\kappa})-i\nu_3)+\alpha_3\alpha_2^*|\Phi_1|^2 = 0.$$

The system is unstable if a disturbance which grows in time exists. Thus we seek that real value of $\mathbf{m}$ which maximizes the growth rate (i.e. minimizes the imaginary part of $s$). After a straightforward but lengthy calculation it can be shown that the appropriate value of $\mathbf{m}$ satisfies the equation

$$\mathbf{m}.(\mathbf{c}_3-\mathbf{c}_2) = -\boldsymbol{\kappa}.(\delta\mathbf{c}_3+\mathbf{c}_2)-\frac{(\nu_2-\nu_3)\mathscr{I}(\alpha_3\alpha_2^*)}{|\alpha_3\alpha_2^*|+\mathscr{R}(\alpha_3\alpha_2^*)}$$

and the maximum growth rate is

$$\tfrac{1}{2}\{[(\nu_2-\nu_3)^2+2|\Phi_1|^2(|\alpha_3\alpha_2^*|+\mathscr{R}(\alpha_3\alpha_2^*))]^{\frac{1}{2}}-(\nu_2+\nu_3)\}, \qquad (7)$$

where $\mathscr{R}$ and $\mathscr{I}$ respectively denote real and imaginary parts. We note that if $\alpha_3\alpha_2^*$ is both real and negative there cannot be any instability.

Until now there have been no restrictions upon the direction of propagation of the long waves. However, from (7) it is clear that the growth rate of the long waves will depend on their direction through the variations of $\nu_3, \alpha_2$ and $\alpha_3$ with $\boldsymbol{\kappa}$. It is natural to expect the waves with the greatest growth rate to dominate. If $\theta$ measures the angle between $\mathbf{k}$ and $\boldsymbol{\kappa}$, then at the direction of greatest growth rate $\theta$ is given implicitly by the formula

$$|\Phi_1|^2\frac{\partial}{\partial\theta}(|\alpha_3\alpha_2^*|+\mathscr{R}(\alpha_3\alpha_2^*))$$
$$+\frac{\partial\nu_3}{\partial\theta}\{\nu_3-\nu_2-[(\nu_2-\nu_3)^2+2|\Phi_1|^2(|\alpha_3\alpha_2^*|+\mathscr{R}(\alpha_3\alpha_2^*))]^{\frac{1}{2}}\} = 0.$$

When the system is only slightly unstable, any anisotropy of the medium with respect to the long waves can cause the direction of greatest growth rate to be significantly changed from the direction that would be predicted from a theory which did not include some allowance for dissipation.

In some circumstances there may be geometrical constraints, such as side walls in a wave tank, which restrict the long waves to one particular value of $\boldsymbol{\kappa}$. In such a case equations (6) provide a means of determining the motion which will develop. In particular, if $B$ is constant then one possible final state is $\Phi_j$ = constant, provided that $\delta$ takes the value

$$\frac{\nu_3(\nu_2\mathscr{I}(\alpha_3\alpha_2^*)-\mathbf{c}_2.\boldsymbol{\kappa}\mathscr{R}(\alpha_3\alpha_2^*))}{\mathbf{c}_3.\boldsymbol{\kappa}(\nu_2\mathscr{R}(\alpha_3\alpha_2^*)+\mathbf{c}_2.\boldsymbol{\kappa}\mathscr{I}(\alpha_3\alpha_2^*))}.$$

Unfortunately it is not clear whether these steady solutions are stable.

## 4. Applicability

From equations (5) and the approximate expressions given in the appendix for the $Q_j$ and $R_j$ it can readily be shown that

$$\alpha_1 = \frac{1}{2D}\left\{\frac{k}{\kappa}\frac{\omega_1+\omega_2}{\omega_1}I\cos\theta+O(1)\right\},$$

$$\alpha_2 = \frac{-1}{2D}\left\{\frac{k}{\kappa}\frac{\omega_1+\omega_2}{\omega_2}I\cos\theta+O(1)\right\},$$

$$\alpha_3 = \frac{-1}{2D}\left\{\frac{k}{\kappa}\frac{\omega_1^2-\omega_2^2}{\omega_1\omega_2}I\cos\theta+O(1)\right\},$$

where $I$ is a constant defined by the integral

$$\int_{-D}^{0} N^2W_1W_2\frac{dW_3}{dz}dz.$$

In practice there are usually two features which simplify the calculations of the functions $W_1, W_2, W_3$. First, for the internal wave modes the surface $z = 0$ is effectively rigid; second, for the surface waves the water is quite shallow. For the instability mechanism this is an unfortunate combination of approximations, since $dW_3/dz$ is constant and the orthogonality of the 'rigid lid' normal modes means that either $\omega_1^2-\omega_2^2$ or $I$ is zero. Thus, in order to get an explicit expression for $\alpha_3\alpha_2^*$ we must calculate some second-order terms. For example,

if the shallow-water approximation is relaxed, then the first approximation for $I$ is
$$\frac{3(\omega_1-\omega_2)^2 D}{g}\int_{-D}^{0} N^2 W_1 W_2\left\{\int_{-D}^{z}\left(1-\frac{N^2}{(\omega_1-\omega_2)^2}\right)\left(\frac{z'+D}{D}\right)\frac{dz'}{D}\right\}\frac{dz}{D},$$
and provided that $\omega_1^2 > \omega_2^2$ the coupling constant $\alpha_3\alpha_2^*$ is non-negative.

In the oceans we can expect $\omega_1, \omega_2$ and $\omega_1-\omega_2$ to be of the order of $N_m$, the maximum value of $N$. Thus, with the above expression for $I$ we can estimate $\alpha_3\alpha_2^*$ as being of the order $k^2N_m^4/g^2\kappa^2$. The results of Le Blond (1966) show that $\nu_2$ is of the order $10^{-3}N_m$. Since $\nu_3$ is negligible relative to $\nu_2$ we can deduce from the above estimates and from (7) that for the onset of instability $\Phi_1$ must be of the order $10^{-3}g\kappa/N_m k$, or equivalently that the internal wave must have a displacement amplitude of the order $10^{-3}g\kappa/N_m^2 k$. For the deep ocean $N_m$ is characteristically $5\times 10^{-3}\,\mathrm{s}^{-1}$ so the corresponding critical displacement amplitude is $400\kappa/k$m, which can be small because the factor $\kappa/k$ is small. It should be noted that these estimates only apply when both the 'rigid lid' and the $\kappa/k \ll 1$ approximations are more accurately satisfied than the shallow-water approximation.

## 5. Discussion

When one examines the structure of the calculations in §§2 and 3, it becomes clear that the value of the coupling constant $\alpha_3\alpha_2^*$ is the only factor that depends upon the detailed physics of the situation. Equations of the form (6) are to be expected in any system with weak quadratic nonlinearities in which there are two classes of waves with greatly different wavelengths. Such systems are by no means rare: in the oceans, for example, baroclinic-Rossby, Kelvin, edge and continental-shelf waves all have barotropic counterparts with much greater wavelengths. Unfortunately, because there is the one exceptional case in which the coupling constant $\alpha_3\alpha_2^*$ is real and negative, there may be some circumstances where the resonant triad mechanism does not lead to the generation of long waves.

## Appendix

Using the same notation as Phillips (1966, chapter 5) the expressions for $Q$ and $R$, correct to second order in the wave amplitude, are
$$Q = \frac{\partial^2}{\partial z\,\partial t}\nabla_h.(\mathbf{u}.\nabla\mathbf{q}) - \nabla_h^2\left\{\mathbf{u}.\nabla b + \frac{\partial}{\partial t}(\mathbf{u}.\nabla w)\right\}$$
and
$$R = \frac{\partial}{\partial t}\nabla_h.(\mathbf{u}.\nabla\mathbf{q}) - \nabla_h^2\left\{\frac{\partial}{\partial t}\left(\tfrac{1}{2}N^2\zeta^2 + \frac{1}{\rho_0}\frac{\partial p}{\partial z}\zeta\right) + g\nabla_h.(\mathbf{q}\zeta)\right\}.$$
For a wave of the form
$$w = W(z;\mathbf{m},n)\phi\exp[int + i\mathbf{m}.\mathbf{x}]$$
the expressions for the other symbols are
$$\mathbf{q} = i\frac{dW}{dz}\frac{\mathbf{m}}{m^2},\quad \mathbf{u} = \mathbf{q} + W\hat{\mathbf{z}},$$

$$b = iN^2 \frac{W}{n}, \quad \frac{1}{\rho_0}\frac{\partial p}{\partial z} = i(N^2 - n^2)\frac{W}{n},$$

$$\zeta = -i\frac{W}{n}\bigg|_{z=0},$$

where in each expression the factor $\phi \exp[int + i\mathbf{m}.\mathbf{x}]$ has been suppressed.

Using the above equations it is straightforward to evaluate the functions $Q_j$ and $R_j$ which were introduced in equations (4). The full expressions for these functions are simplified considerably if we exploit the facts that $\kappa/k \ll 1$ and that $(\omega_1 - \omega_2)^2/g\kappa$ is of order 1. The simplified expressions are

$$Q_1 = i\Phi_2\Phi_3\left\{-\frac{k^3}{\kappa}\cos\theta\frac{\omega_1+\omega_2}{\omega_2^2}N^2W_2\frac{dW_3}{dz} + O(k^2)\right\},$$

$$R_1 = i\Phi_2\Phi_3\frac{dW_2}{dz}\frac{dW_3}{dz}\bigg|_{z=0}\left\{\frac{k^2}{\kappa^2}(\omega_1-\omega_2) + O(k/\kappa)\right\},$$

$$Q_2 = i\Phi_1\Phi_3^*\left\{\frac{k^3}{\kappa}\cos\theta\frac{\omega_1+\omega_2}{\omega_1^2}N^2W_1\frac{dW_3}{dz} + O(k^2)\right\},$$

$$R_2 = i\Phi_1\Phi_3^*\frac{dW_1}{dz}\frac{dW_3}{dz}\bigg|_{z=0}\left\{-\frac{k^2}{\kappa^2}(\omega_1-\omega_2) + O(k/\kappa)\right\},$$

$$Q_3 = i\Phi_1\Phi_2^*\left\{-k\kappa\cos\theta\frac{(\omega_1-\omega_2)(\omega_1^2-\omega_2^2)}{\omega_1^2\omega_2^2}\frac{d}{dz}(N^2W_1W_2) + O(\kappa^2)\right\},$$

$$R_3 = i\Phi_1\Phi_2^*\frac{dW_1}{dz}\frac{dW_2}{dz}\bigg|_{z=0}\left\{\frac{\kappa^2}{k^2}\cos 2\theta(\omega_1-\omega_2) + O(\kappa^3/k^3)\right\}.$$

We note that if the higher order terms are required then it is necessary to use the exact dispersion relations in determining $\omega_2$ and $\boldsymbol{\kappa}$.

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